Fairness and Privacy in the Optimal Transport for Resource Allocation

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Abstract

- Optimal transport (OT) is a way of finding the most ***efficient*** way to allocate a limited amount of resources. However, the most efficient transport plan may not always lead to a fair allocation of resources.
- We introduced a framework that incorporates fairness in resource allocation through adding a ***fairness factor*** into the optimal transport plan.
- Using the alternating method of multipliers, we developed a ***distributed algorithm*** that negotiates solutions based on each pair of source and target directly, and it preserves the ***privacy*** of participants.
- The algorithm yields fair and efficient results which are corroborated through case studies.

Motivation

(a)Hurricane's uneven damages to racially different neighborhoods

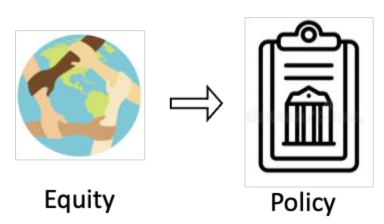
(b) Vaccine distribution among cities, communities, etc.

Figure1: Possible applications of fair optimal transport.

- Fairness and privacy are important factors to consider when allocating a limited amount of resources, particularly in those scenarios with societal impacts.
- The classical OT framework lacks such critical considerations.

Objective

- To implement the OT paradigm with fairness into a distributed algorithm for fair and efficient resource allocation over a network.



Problem Formulation

1. Optimal Transport Framework

$$\max_{\Pi} \sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi_{xy}) + \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y}(s_{xy}(\pi_{xy}) - c_{xy}(\pi_{xy}))$$
$$\text{s.t.}\quad \underline{p}_x \le \sum_{y\in\mathscr{Y}_x}\pi_{xy} \le \bar{p}_x,\ \forall x\in\mathscr{X},$$
$$\underline{q}_y \le \sum_{x\in\mathscr{X}_y}\pi_{xy} \le \bar{q}_y,\ \forall y\in\mathscr{Y},$$
$$\pi_{xy} \ge 0,\ \forall (x,y)\in\mathscr{E},$$

*$\mathcal{X}_y$ is the set of target nodes connected to source **y** and $\mathcal{Y}_x$ is the set of source nodes connected to target **x**.*

$\mathbf{\Pi}$ denotes the set of transport plans, $\{\pi_{xy}\}_{x\in\mathscr{X}_y, y\in\mathscr{Y}}$

2. Fairness Function

$$f_x(\sum_{y\in\mathscr{Y}_x}\pi_{xy}) = \log(\sum_{y\in\mathscr{Y}_x}\pi_{xy} + 1),\ \forall x\in\mathscr{X}.$$

3. Incorporating Fairness into the OT Framework

$$\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi_{xy}) + \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y}(s_{xy}(\pi_{xy}) - c_{xy}(\pi_{xy})) + \sum_{x\in\mathscr{X}}\omega_x f_x(\sum_{y\in\mathscr{Y}_x}\pi_{xy}),$$

$\omega_x \ge 0$ is the weighing constant for fairness

Methodology

Distributed Algorithm

1. Problem Reformulation

Transform the original OT framework into a minimization problem,

$$\min_{\Pi_t\in\mathscr{F}_t,\Pi_s\in\mathscr{F}_s,\Pi} -\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi_{xy}^t) - \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y}(s_{xy}(\pi_{xy}^s) - c_{xy}(\pi_{xy}^s)) - \sum_{x\in\mathscr{X}}\omega_x f_x(\sum_{y\in\mathscr{Y}_x}\pi_{xy}^t)$$
$$\text{s.t.}\quad \pi_{xy}^s = \pi_{xy},\ \forall (x,y)\in\mathscr{E},$$
$$\pi_{xy}^t = \pi_{xy},\ \forall (x,y)\in\mathscr{E},$$

2. Distributed Algorithm

After simplifying and applying ADMM by using the Lagrangian we have,

$$\Pi_{x,t}(k+1) \in \arg\min_{\Pi_{x,t}\in\mathscr{F}_{x,t}} -\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi_{xy}^t) - \omega_x f_x(\sum_{y\in\mathscr{Y}_x}\pi_{xy}^t) + \sum_{y\in\mathscr{Y}_x}\alpha_{xy}(k)\pi_{xy}^t + \frac{\eta}{2}\sum_{y\in\mathscr{Y}_x}\left(\pi_{xy}^t - \pi_{xy}(k)\right)^2,$$

$$\Pi_{y,s}(k+1) \in \arg\min_{\Pi_{y,s}\in\mathscr{F}_{y,s}} -\sum_{x\in\mathscr{X}_y}\left(s_{xy}(\pi_{xy}^s) - c_{xy}(\pi_{xy}^s)\right) - \sum_{x\in\mathscr{X}_y}\alpha_{xy}(k)\pi_{xy}^s + \frac{\eta}{2}\sum_{x\in\mathscr{X}_y}\left(\pi_{xy}(k) - \pi_{xy}^s\right)^2,$$

$$\pi_{xy}(k+1) = \frac{1}{2}\left(\pi_{xy}^t(k+1) + \pi_{xy}^s(k+1)\right),$$

$$\alpha_{xy}(k+1) = \alpha_{xy}(k) + \frac{\eta}{2}\left(\pi_{xy}^t(k+1) - \pi_{xy}^s(k+1)\right).$$

α_{xy} is the Lagrangian multiplier

3. Summary

Algorithm 1 Distributed Algorithm

while $\Pi_{x,t}$ and $\Pi_{y,s}$ not converging **do**
  Compute $\Pi_{x,t}(k+1)$ using (13), for all $x \in \mathscr{X}_y$
  Compute $\Pi_{y,s}(k+1)$ using (14), for all $y \in \mathscr{Y}_x$
  Compute $\pi_{xy}(k+1)$ using (15), for all $\{x,y\} \in \mathscr{E}$
  Compute $\alpha_{xy}(k+1)$ using (16), for all $\{x,y\} \in \mathscr{E}$
end while
return $\pi_{xy}(k+1)$, for all $\{x,y\} \in \mathscr{E}$

Case Study I

Fairness Factor and Social Utility

- In the distributed algorithm, fairness is described as the term $\sum_{x\in\mathscr{X}}\omega_x f_x(\sum_{y\in\mathscr{Y}_x}\pi_{xy})$ where f_x is a function that takes the log of the transport plan and ω is a weighing factor.
- To see the different effects of ω, different values were tested with a constant number or source and target nodes. When ω increases, social utility converges at a greater point.
- This indicates that if fairness is more preferred than efficiency, a larger ω should be used.
- The two graphs seem to converge at similar iterations but at significantly different utilities.
- This shows that although ω does not affect convergence rate, it does affect social utility and efficiency.

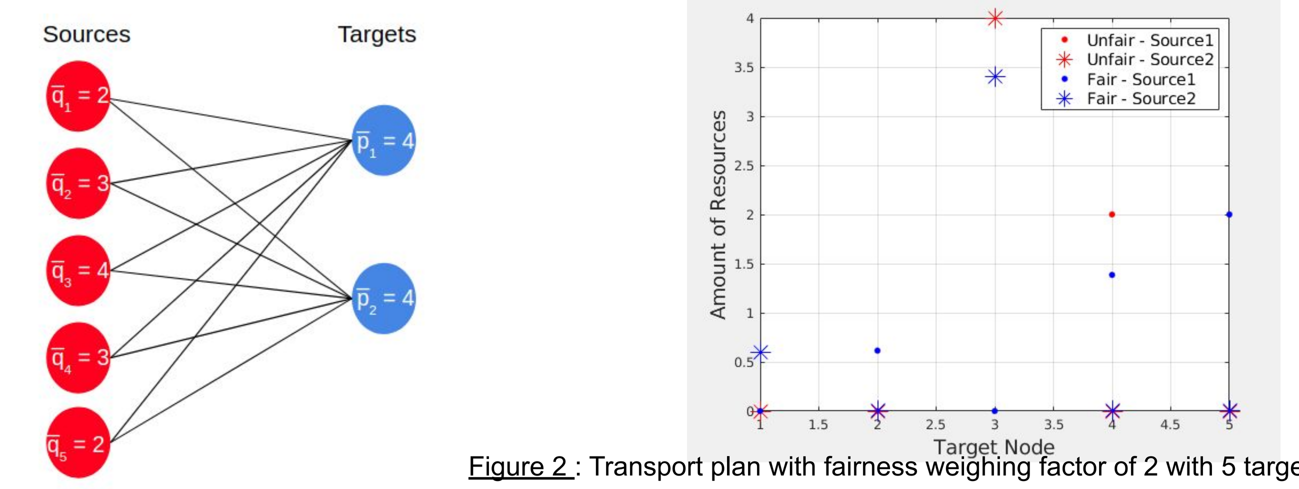


Figure 2 : Transport plan with fairness weighing factor of 2 with 5 targets and 2 sources.

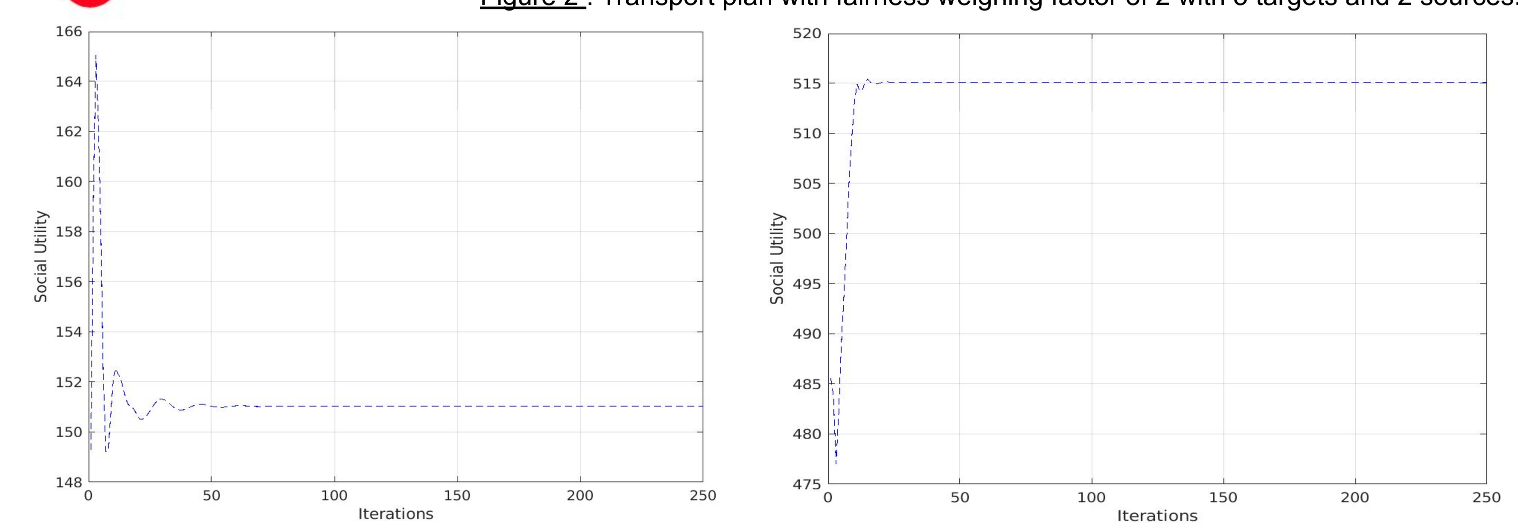


Figure 3 : Social utility of transport plan with weighing factor of 2 (left) and 20 (right) with 7 targets and 5 sources.

Case Study II

Online Distributed Algorithm

- To implement the distributed algorithm to an online case, we consider changes in resource allocation over time with the entrance and exits of target nodes and source nodes and changes in connection schemes over intervals of 100 iterations.
- We see that the algorithm still converges to the centralized optimal solution (the black line in Figure 5(a)) and it is consistent with the distributed algorithm.

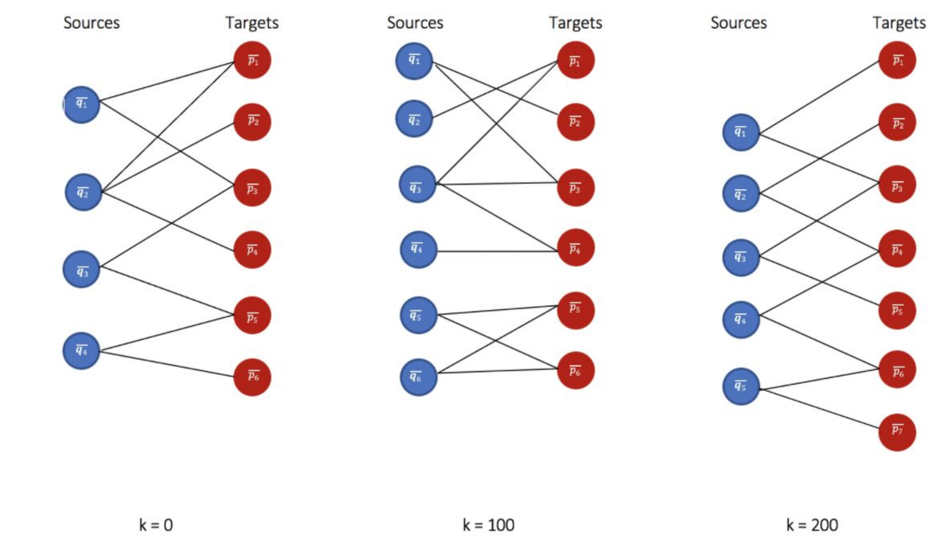
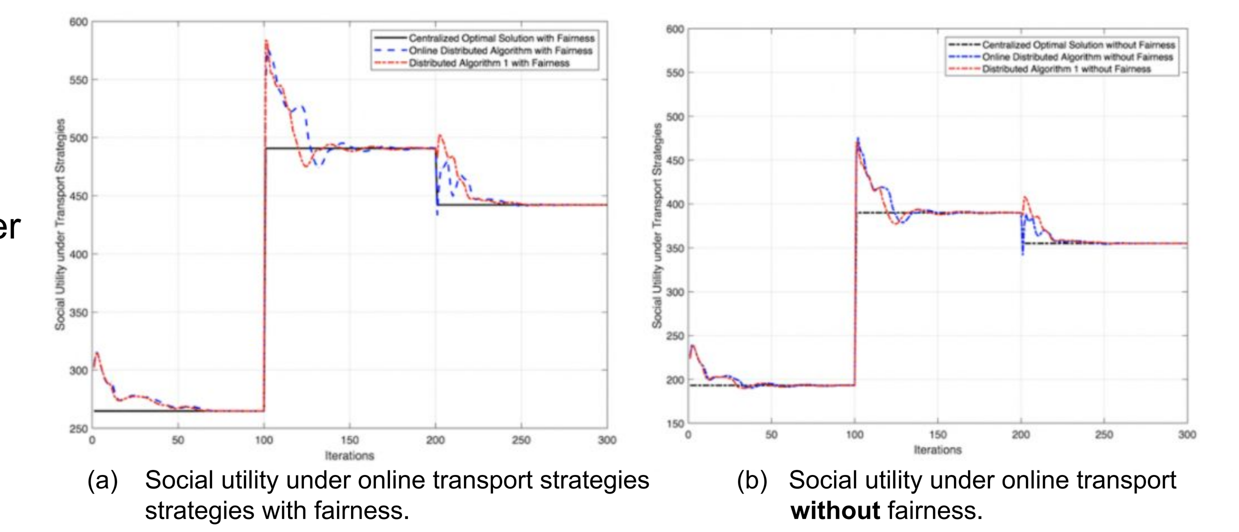


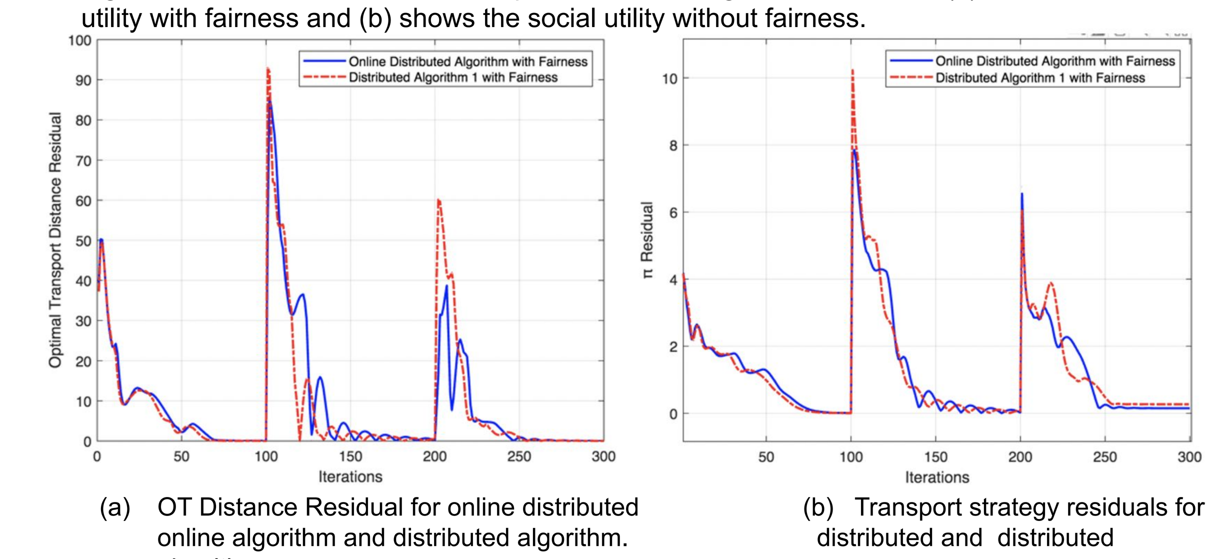
Figure 4: Network structures for the online resource distribution. (Note that upper bounds are not defined since we will be using random values).



(a) Social utility under online transport strategies strategies with fairness.

(b) Social utility under online transport **without** fairness.

Figure 5: The network structure and preferences change at k = 100,200. (a) shows the social utility with fairness and (b) shows the social utility without fairness.



(a) OT Distance Residual for online distributed online algorithm and distributed algorithm.

(b) Transport strategy residuals for distributed and distributed

Figure 6: The residuals for the online and the distributed algorithm are consistent.

Conclusion

- The effect of fairness in the efficient transport of finite resources was investigated in this study.
- Through the distributed algorithm we were able to allow participants to update their own preferences but maintain the same transport plan as the centralized algorithm.
- In the case studies, we can see that the fairness term promotes negotiations between the source and target nodes which leads to more social utility.
- The online distributed resource allocation also allows for the problem to be continuously solved while the scenario is changed.
- This inspires future work in applying the distributive algorithm to a real life situation like COVID-19 vaccine distribution.